

LARGE SUBPOSETS WITH SMALL DIMENSION

BENJAMIN REINIGER* AND ELYSE YEAGER

ABSTRACT. Dorais asked for the maximum guaranteed size of a dimension d subposet of an n -element poset. A lower bound of order \sqrt{n} was found by Goodwillie. We provide a sublinear upper bound for each d . For $d = 2$, our bound is $n^{0.8295}$.

1. INTRODUCTION

Given a family of posets \mathcal{F} , let $\text{ex}^*(P, \mathcal{F})$ denote the size of the largest induced subposet of P that does not contain any member of \mathcal{F} as an induced subposet. Similarly, $\text{ex}(P, \mathcal{F})$ is the size of the largest induced subposet of P that does not contain a member of \mathcal{F} as a not-necessarily-induced subposet. These can be seen as poset analogues of the *relative Turán* numbers of families of graphs (in some host graph). We write $\text{ex}^*(P, \{Q\})$ as simply $\text{ex}^*(P, Q)$. Let $\text{ex}^*(n, \mathcal{F})$ denote the minimum of $\text{ex}^*(P, \mathcal{F})$ over all n -element posets P . In other words, $\text{ex}^*(n, \mathcal{F})$ is the maximum k such that every n -element poset P has an \mathcal{F} -free subposet of size at least k . Let B_n be the boolean lattice of dimension n and A_n an antichain on n points.

Then $\text{ex}^*(P, B_1)$ is just the width of P and $\text{ex}^*(P, A_2)$ is the height of P . The function $\text{ex}(B_n, B_2)$ is heavily studied as the maximum size of a “diamond-free” family of sets. In the literature, $\text{ex}(B_n, P)$ is denoted $\text{La}(n, P)$, and $\text{ex}^*(B_n, P)$ is denoted $\text{La}^\#(n, P)$ or $\text{La}^*(n, P)$.

In this note we are concerned with finding large subposets of small dimension. Hence we let \mathcal{D}_d denote the family of posets of dimension at least d , and ask

Question 1.1. *What is $\text{ex}^*(n, \mathcal{D}_{d+1})$?*

In other words, what is the largest size of a dimension d subposet we are guaranteed to find in an n -element poset? (Note that when $d = 1$, A_n shows that $\text{ex}^*(n, \mathcal{D}_{d+1}) = 1$. We henceforth assume $d > 1$.) This question was originally posed by F. Dorais [2], whose aim was to eventually understand the question for infinite posets [1]. Goodwillie [4] proved that $\text{ex}^*(n, \mathcal{D}_{d+1}) \geq \sqrt{dn}$ by considering the width of P : if $w(P) \geq \sqrt{dn}$, then a maximum antichain is a large subposet of dimension 2; if $w(P) \leq \sqrt{dn}$, then by Dilworth’s theorem the union of some d chains has $\geq \sqrt{dn}$ elements, and this has dimension at most d .

We provide a sublinear upper bound by considering the lexicographic power of standard examples. Theorem 2.1 finds the extremal number for lexicographic powers, and Corollary 2.2 applies this to $\text{ex}^*(n, \mathcal{D}_3)$. For other d , Table 1 provides upper bounds on $\text{ex}^*(n, \mathcal{D}_{d+1})$.

*corresponding author; email: reinige1@illinois.edu
 reinige1@illinois.edu, yeager2@illinois.edu
 Mathematics Dept., University of Illinois, Urbana-Champaign.

2. MAIN THEOREM

Given a poset P and positive integer k , let P^k denote the lexicographic order on k -tuples of elements of P .

Theorem 2.1. *Let P be a poset, \mathcal{F} a family of posets, k a positive integer, and let $n = |P|^k = |P^k|$. Then $\text{ex}^*(|P|^k, \mathcal{F}) \leq \text{ex}^*(P^k, \mathcal{F}) \leq n^{\log_{|P|}(\text{ex}^*(P, \mathcal{F}))}$.*

Proof. Let S be a maximum \mathcal{F} -free subposet of P^k (so $|S| = \text{ex}^*(P^k, \mathcal{F})$). For $i \leq k + 1$ and each i -tuple α , let

$$\begin{aligned} S_\alpha &= \{s \in S : \alpha \text{ is an initial segment of } s\}, \\ Q(\alpha) &= \{p \in P : (\alpha, p) \text{ is an initial segment of some } s \in S\}. \end{aligned}$$

Then each $Q(\alpha)$ is an induced subposet of S , under any of the maps that assign to $p \in P$ an element $s \in S$ with initial segment (α, p) . Since S is \mathcal{F} -free, so is $Q(\alpha)$, hence $|Q(\alpha)| \leq \text{ex}^*(P, \mathcal{F})$.

We have that

$$|S_\alpha| = \sum_{p \in Q(\alpha)} |S_{(\alpha, p)}| \leq |Q(\alpha)| \cdot \max_{p \in Q(\alpha)} |S_{(\alpha, p)}| \leq \text{ex}^*(P, \mathcal{F}) \cdot \max_{p \in Q(\alpha)} |S_{(\alpha, p)}|.$$

When ω is a k -tuple, S_ω is either $\{\omega\}$ or \emptyset . Hence we have, for α an i -tuple,

$$|S_\alpha| \leq (\text{ex}^*(P, \mathcal{F}))^{k-i},$$

and in particular, for α the 0-tuple,

$$|S| \leq (\text{ex}^*(P, \mathcal{F}))^k = |P|^{\log_{|P|}(\text{ex}^*(P, \mathcal{F}))^k} = n^{\log_{|P|}(\text{ex}^*(P, \mathcal{F}))}.$$

□

Corollary 2.2. *For all sufficiently large n , $\text{ex}^*(n, \mathcal{D}_3) \leq n^{0.8295}$.*

Proof. Take $P = S_m$, the standard example on $2m$ points, in the preceding theorem. It is easy to see that $\text{ex}^*(S_m, \mathcal{D}_3) = m + 2$. Hence the exponent on the family of posets obtained is $\log_{2m}(m + 2)$, which is minimized at $m = 10$ with value approximately 0.82948. This completes the proof when n is a power of 20.

Otherwise, write $n = \sum_{i=0}^k \alpha_i (20)^i$, each $\alpha_i \in \{0, \dots, 19\}$. Then let Q be the poset that is the disjoint union of α_i copies of S_{10}^i for each i . A maximum dimension 2 subposet of Q is precisely the union of maximum dimension 2 subposets of each

S_{10}^i . So

$$\begin{aligned}
\text{ex}^*(n, \mathcal{D}_3) &\leq \text{ex}^*(Q, \mathcal{D}_3) \\
&= \sum_{i=0}^k \alpha_i \text{ex}^*(S_{10}^i, \mathcal{D}_3) \\
&\leq \sum_{i=0}^k \alpha_i (20)^{0.82949i} \\
&\leq \left(\sum_{i=0}^k \alpha_i \right) \left(\frac{\sum_{i=0}^k \alpha_i (20)^i}{\sum_{i=0}^k \alpha_i} \right)^{0.82949} \quad (\text{Jensen's inequality}) \\
&= \left(\sum_{i=0}^k \alpha_i \right)^{1-0.82949} n^{0.82949} \\
&\leq (19(\lfloor \log_{20} n \rfloor + 1))^{0.17051} n^{0.82949} \\
&< n^{0.8295}
\end{aligned}$$

for sufficiently large n . \square

Essentially the same proof works for any d . We have for any m and any $\epsilon > 0$ that for sufficiently large n , $\text{ex}^*(n, \mathcal{D}_{d+1}) \leq n^{\log_{2m}(m+d)+\epsilon}$. Table 1 shows some values of d with the minimizing m and the minimum value of the exponent (rounded to the 5th decimal place).

d	m	$\log_{2m}(m+d)$
2	10	0.82948
3	17	0.84953
4	25	0.86076
10	78	0.88663
100	1169	0.92122

TABLE 1. Values of m that minimize $\log_{2m}(m+d)$ for given d .

3. REMARKS

There is still a rather large gap between the known lower and upper bounds for $\text{ex}^*(n, \mathcal{D}_{d+1})$. Any improvement to either the lower or upper bound would be interesting.

Given the interest in $\text{ex}(B_n, B_2)$, one may be interested in $\text{ex}^*(B_n, \mathcal{D}_{d+1})$ instead of $\text{ex}^*(n, \mathcal{D}_{d+1})$.

Question 3.1. *What is $\text{ex}^*(B_n, \mathcal{D}_{d+1})$?*

Lu and Milans (personal communication) have shown that $\text{ex}^*(B_n, S_d) \leq (4d + C\sqrt{d} + \epsilon) \binom{n}{\lfloor n/2 \rfloor}$. Hence also $\text{ex}^*(B_n, \mathcal{D}_d) = \Theta(\binom{n}{\lfloor n/2 \rfloor})$. For small cases, we have computed that $\text{ex}^*(B_n, \mathcal{D}_3) = 1, 4, 7, 12, 20$ for $n = 1, 2, 3, 4, 5$.

In 1974, Erdős [3] posed and partially answered the following question: given an r -uniform hypergraph $G_r(n)$ on n vertices such that every m -vertex subgraph has

chromatic number at most k , how large can the chromatic number of $G_r(n)$ be? Using probability methods Erdős found a lower bound for ordinary graphs when $k = 3$; that is, when every m -vertex subgraph has chromatic number at most 3. Thinking of poset dimension as analogous to graph chromatic number, we ask:

Question 3.2. *Given a poset P with n elements such that every m -element subposet has dimension at most d , how large can the dimension of P be?*

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